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Monte Carlo study of the $Z(5)$ model†

V L V Baltar, G M Carneiro, M E Pol and N Zagury

Departamento de Física, Pontifícia Universidade Católica, Cx.P. 38071, Rio de Janeiro, RJ, Brasil

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Abstract. The phase diagram of the $Z(5)$ model on a square lattice is studied using Monte Carlo simulation techniques. We confirm the existence of three phases in the ferromagnetic $Z(5)$ model and find a critical phase for the antiferromagnetic model. We determine the phase boundaries using Monte Carlo renormalisation group methods.

1. Introduction

The $Z(5)$ model is defined as follows. Consider a square lattice whose sites are occupied by classical 'spins', S, which can take values $e^{i\theta}$, $\theta = 2k\pi/5$, $k = 0, 1, \ldots, 4$. Assuming nearest-neighbour interactions only, the total energy of the $Z(5)$ model on a square lattice is defined as

$$H = -J_1 \sum_{\boldsymbol{n},\boldsymbol{\mu}} \{\cos[\theta(\boldsymbol{n}) - \theta(\boldsymbol{n}+\boldsymbol{\mu})] - 1\} - J_2 \sum_{\boldsymbol{n},\boldsymbol{\mu}} \{\cos 2[\theta(\boldsymbol{n}) - \theta(\boldsymbol{n}+\boldsymbol{\mu})] - 1\}. \tag{1}$$

In equation (1) $\boldsymbol{n}$ is a vector that labels the lattice sites, $\boldsymbol{\mu}$ represents the conventional primitive vectors of the square lattice, and J_1 and J_2 are the coupling constants.

According to equation (1) there are, in general, three distinct possibilities for the energy of a pair of nearest-neighbour spins, depending on the angle $\Delta\theta$ between them:

$$\begin{aligned} &E_0 = 0 && \Delta\theta = 0 \\ &E_1 = aJ_1 + bJ_2 && \Delta\theta = \pm 2\pi/5 \\ &E_2 = bJ_1 + aJ_2 && \Delta\theta = \pm 4\pi/5 \end{aligned} \tag{2}$$

where

$$a = 1 - \cos 2\pi/5 = \frac{\sqrt{5}}{4}(\sqrt{5} - 1) \simeq 0.691$$

$$b = 1 - \cos 4\pi/5 = \frac{\sqrt{5}}{4}(\sqrt{5} + 1) \simeq 1.809.$$

From equations (2) we see that E_1 and E_2 are doubly degenerate and that the partition function $\mathscr{Z} = \mathrm{Tr}\, e^{-H/kT}$ is invariant under the exchange of J_1 and J_2.

Throughout this paper we refer to the region of the coupling constant space where both E_1 and E_2 are positive as the ferromagnetic (F) region and to that where at least one of them is negative as the antiferromagnetic (AF) region.

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In terms of J_1 and J_2 the F region corresponds to

$$J_1 > 0 \qquad J_2/J_1 > -a/b \simeq -0.382$$

$$J_1 < 0 \qquad J_2/J_1 < -b/a \simeq -2.618$$

and the AF region corresponds to

$$J_1 > 0 \qquad J_2/J_1 < -a/b \simeq -0.382$$

$$J_1 < 0 \qquad J_2/J_1 > -b/a \simeq -2.618.$$

It can be shown that $\mathscr{Z}$ obeys the generalised self-duality relations (Alcaraz and Köberle 1980):

$$\mathscr{Z}(x_1, x_2) = B\mathscr{Z}(x_1^*, x_2^*) \tag{3}$$

where

$$\begin{aligned} x_1 &= \exp(-E_1/kT) \\ x_2 &= \exp(-E_2/kT) \end{aligned} \tag{4}$$

and

$$\begin{aligned} x_1^* &= \frac{1+2(1-a)x_1+2(1-b)x_2}{1+2x_1+2x_2} \\ x_2^* &= \frac{1+2(1-b)x_1+2(1-a)x_2}{1+2x_1+2x_2} \end{aligned} \tag{5}$$

and B is a non-singular function of J_1/kT and J_2/kT. Generalised self-duality holds only if

$$\begin{aligned} (1-a)x_1 + (1-b)x_2 &> -\tfrac{1}{2} \\ (1-b)x_1 + (1-a)x_2 &> -\tfrac{1}{2} \end{aligned} \tag{6}$$

since both x_1^* and x_2^* should be positive.

The region of the coupling constant space defined by equations (6), which we refer to as the duality (D) region, lies within the F region.

The phase diagram of the $Z(5)$ model in the F region has been studied by many authors (Elitzur *et al* 1979, Alcaraz and Köberle 1980, Einhorn *et al* 1980, Domany *et al* 1980, Rujan *et al* 1981, Roomany and Wyld 1981). Elitzur *et al* (1979) and Alcaraz and Köberle (1980), using duality and Griffiths inequalities between correlation functions, suggested the existence in the F region of a critical phase, characterised by power-law decay of correlations, besides the usual ferromagnetic and disordered ones. Domany *et al* (1980) obtained a critical phase for the $Z(5)$ model on a square lattice using Monte Carlo simulations and exploring the mapping of the $Z(5)$ model on the SOS model along the $x_1 = 0$ (or $x_2 = 0$) line. The existence of a critical phase in the F region is further supported by Monte Carlo renormalisation group studies carried out by Tobochnick (1982) for the five-state ferromagnetic clock model, which is a particular case of the $Z(5)$ model with $J_2 = 0$. However, calculations of the phase diagram using the Migdal–Kadanoff renormalisation group transformation, carried out by Rujan *et al* (1981), and finite-size scaling, carried out by Roomany and Wyld (1981), failed to obtain a critical phase. Our results, described in § 2, further confirm the existence of a critical phase in the F region.

In the AF region much less is known about the phase diagram. An interesting question is whether there is a critical phase in this region or not.

The ground state in the AF region is such that nearest-neighbour spins make an angle of $\pm\frac{2}{5}\pi$ (or $\pm\frac{4}{5}\pi$) between them. The degeneracy of this ground state gives rise to extensive ground-state entropy. Berker and Kadanoff (1980) suggested that systems with this property might exhibit a critical phase. Baxter (1982) showed that the AF N-state Potts model defined on a square lattice, which also has extensive ground-state entropy, is disordered at all finite temperatures for $N \geq 3$. Thus the five-state Potts model, which is a particular case of the $Z(5)$ model for $J_2/J_1 = 1$ and $J_1 < 0$, does not exhibit a critical phase. Cardy (1981) suggested that the AF N-state clock model belongs to the same universality class of the ferromagnetic $2N$-state clock model, which for $N \geq 3$ has three phases (Elitzur *et al* 1979, Alcaraz and Köberle 1980), one of them being critical. Thus the AF five-state clock model, which is a particular case of the $Z(5)$ model for $J_2 = 0$, $J_1 < 0$, should, according to Cardy's suggestion, belong to the same universality class as the ferromagnetic ten-state clock model. However, studies of the AF three-state Potts model, which is identical to the AF three-state clock model, carried out by den Nijs *et al* (1982), showed that a critical phase occurs only when second-neighbour interactions are considered. Thus it is not clear whether Cardy's suggestion applies to the AF five-state clock model with nearest-neighbour interactions only.

In this paper we investigate the question of the existence of critical phases in the $Z(5)$ model using Monte Carlo (MC) simulation techniques. We analyse the MC data using both the standard technique of following the temperature dependence of thermodynamic quantities, and MC renormalisation group (MCRG) methods, as described in § 2. The combination of these two methods allows us to identify critical phases and to locate, approximately, their boundaries. We confirm the existence of a critical phase in the F region and find evidence that it extends itself into the AF region. The results of this analysis are described in § 3. A summary of our conclusions is presented in § 4.

2. Monte Carlo simulations

In order to study the phase diagram of the $Z(5)$ model over the entire parameter space we carried out Monte Carlo (MC) computer simulations (Binder 1979) on square lattices of sizes ranging from 16×16 to 64×64, subjected to periodic boundary conditions. The spin-flipping procedure used here is similar to that described in a previous paper (Carneiro *et al* 1982).

In this section we describe the method of analysis of the MC data generated in these simulations. Both standard MC methods (Binder 1979) and MC renormalisation group (MCRG) techniques (Swendsen 1982) are discussed.

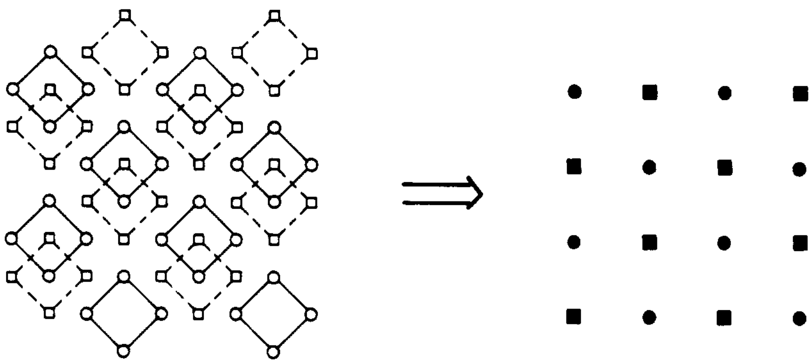
Most of the data was obtained in 20×20 and 40×40 lattices. In these cases standard MC methods of analysis were employed. We calculated the internal energy, specific heat (from the fluctuations of the internal energy), order parameters and the variation with distance of the spin–spin correlation function. Peaks in the specific heat as a function of temperature were used as a first indication of a phase transition. The temperature dependence of the order parameter was used to identify phases and to locate their boundaries. The variation with distance of the spin–spin correlation function was also used to detect phase transitions and to identify possible critical phases.

In these MC simulations, data was taken after 1 MC step/spin. The total number of MC steps/spin used varied from 2500, far from transition points, to 10 000 close to the phase boundaries. Between 30–50% of the initial steps were discarded to allow for thermal equilibrium to be reached.

We also carried out a MCRG analysis of data taken on 16×16, 32×32 and 64×64 lattices and 18×18 and 54×54 lattices, at selected points of the parameter space where the previous analysis suggested the existence of critical phases. The MCRG method used here consists, as usual (Swendsen 1982), of applying a chosen RG transformation to spin configurations generated by the MC simulation.

Spin configurations selected for analysis were recorded every 20 MC steps/spin. The total number of MC steps/spin used to determine each point was 60 000. Close to 20% of the initial steps were discarded to allow for thermal equilibrium to be reached.

Two different RG transformations were used in this analysis. The first one, shown schematically in figure 1, has scale factor $\lambda = 2$. The cells consist of four spins, belonging to the same sublattice of the square lattice, which are second neighbour to each other. The cell spin is determined by the majority rule, using the spin in the left-hand corner as tie-breaker.



(*a*) (*b*)

Figure 1. Schematic representation of the renormalisation group transformation with $\lambda = 2$. The two sublattices are represented by □ and ○, respectively. The four-spin cells are represented by squares in (*a*). The cell spins are represented by ■ and ● in (*b*).

The other RG transformation, shown schematically in figure 2, has scale factor $\lambda = 3$. The cells consist of nine spins, belonging to the same sublattice of the square lattice and the cell spin is also determined by the majority rule.

These RG transformations preserve the symmetry of ground states with ferromagnetic as well as antiferromagnetic order. This means, in general, that states in which the two sublattices are occupied by equal or different spins will be mapped into themselves by the transformations. With this choice of the RG transformation we expect to guarantee that phases with antiferromagnetic order, if present in the phase diagram, are not missed in the analysis, as they probably would if we choose a cell with spins of both sublattices (Niemeijer and van Leeuwen 1976).

To determine the phase boundaries we adopted the method suggested by Wilson. It consists of comparing correlation functions calculated in two $r \times r$ lattices which

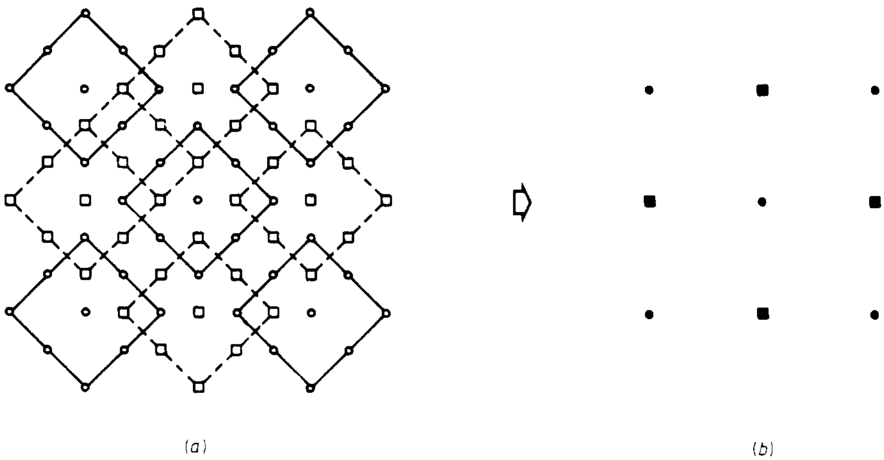


Figure 2. Schematic representation of the renormalisation group transformation with $\lambda = 3$. The two sublattices are represented by □ and ○, respectively. The nine-spin cells are represented by squares in (*a*). The cell spins are represented by ■ and ● in (*b*).

are obtained from original lattices of sizes $L\times L$ and $\lambda L\times\lambda L$ by the application of the RG transformation m and $m+1$ times, respectively ($\lambda^m = L/r$, $\lambda = 2, 3$). The points of the parameter space where the two correlation functions are equal correspond to phase boundaries. If the correlation functions are equal over a finite portion of the parameter space, then this region corresponds to a critical phase (Tobochnik 1982).

In the case of the $Z(5)$ model we calculated first, second and third neighbour correlation functions such as

$$F_p(r, L) = \sum_{\boldsymbol{n},\boldsymbol{\mu}(p)} \langle\cos[\theta(\boldsymbol{n}) - \theta(\boldsymbol{n}+\boldsymbol{\mu}(p))]\rangle \tag{7}$$

and

$$G_p(r, L) = \sum_{\boldsymbol{n},\boldsymbol{\mu}(p)} \langle\cos 2[\theta(\boldsymbol{n}) - \theta(\boldsymbol{n}+\boldsymbol{\mu}(p))]\rangle \tag{8}$$

where $p = 1$, 2 and 3 indicates first, second and third neighbour sites, respectively, and $\boldsymbol{\mu}(p)$ are, respectively, the primitive vectors of the square lattices formed by first neighbour sites ($p = 1$), second neighbour sites ($p = 2$) and third neighbour sites ($p = 3$). Here we work with $r = 4$ and 8, $L = 64$, 32 and 16 or $r = 6$, $L = 54$ and 18.

3. Results

To obtain the phase diagram of the $Z(5)$ model we proceeded as follows. First, we used specific heat peaks as an indication of the points where a phase transition might occur. Our results are shown in figure 3.

At some fixed values of J_2/J_1, suggested by figure 3, we carried out a detailed analysis, using the methods outlined above, to verify if the specific heat peaks do indeed correspond to phase transitions and to identify the phases. The results of this analysis are discussed next.

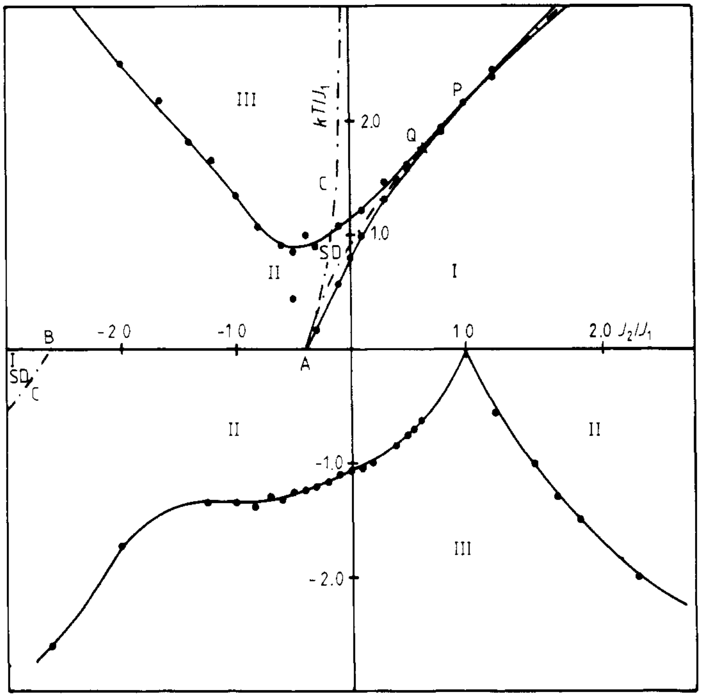


Figure 3. Plot of the location of specific heat peaks. Points represented by ● are the result of MC simulations and those obtained by application of the $J_1 \leftrightarrow J_2$ symmetry. P is the transition point on the scalar Potts model line. Q is the transition point obtained by Fateev and Zamolodchikov. Curve SD is the self-dual line; self-duality holds to the right (left) of curve C for $J_1 > 0$ ($J_1 < 0$). I is the ferromagnetic phase, II is the critical phase and III is the disordered phase. $J_2/J_1 = 0$ is the vector Potts model line. Ferromagnetic ground states are located to the right of point A for $J_1 > 0$, and to the left of point B for $J_1 < 0$.

First we consider the ferromagnetic region $J_1 > 0$ and $J_2/J_1 > -a/b \simeq -0.382$. In this region generalised self-duality holds to the right of curve C in figure 3. We concentrate our discussion on the portion of this region extending from $J_2/J_1 = -0.382$ to $J_2/J_1 \leqslant 1.0$, where, for a fixed value of J_2/J_1, the specific heat against temperature plot has two peaks, indicating the existence of three phases. A region with the same phase structure as that exists for $J_2/J_1 \geqslant 1$, since the phase diagram is symmetric with respect to $J_2 \leftrightarrow J_1$.

A typical curve for the variation of the specific heat, C/Nk, with temperature in this region is shown in figure 4. Also shown is the variation of the order parameter $\langle S \rangle$ with temperature. At all points between $J_2/J_1 = -0.382$ and $J_2/J_1 = 0.8$ shown in figure 3, we observed a similar behaviour for C/Nk and $\langle S \rangle$. The specific heat peaks might indicate two phase transitions, one from the ferromagnetic phase at low temperatures to an intermediate phase and another from this intermediate phase to the disordered phase. According to the arguments put forth by Alcaraz and Köberle (1980) and Elitzur *et al* (1979), if an intermediate phase exists containing the self-dual line it must be critical. The slow variation of $\langle S \rangle$ from ~ 1 to ~ 0, between the two specific heat peaks (figure 4), is not inconsistent with a critical phase, since the finite size of the system allows for a non-zero order parameter in this phase.

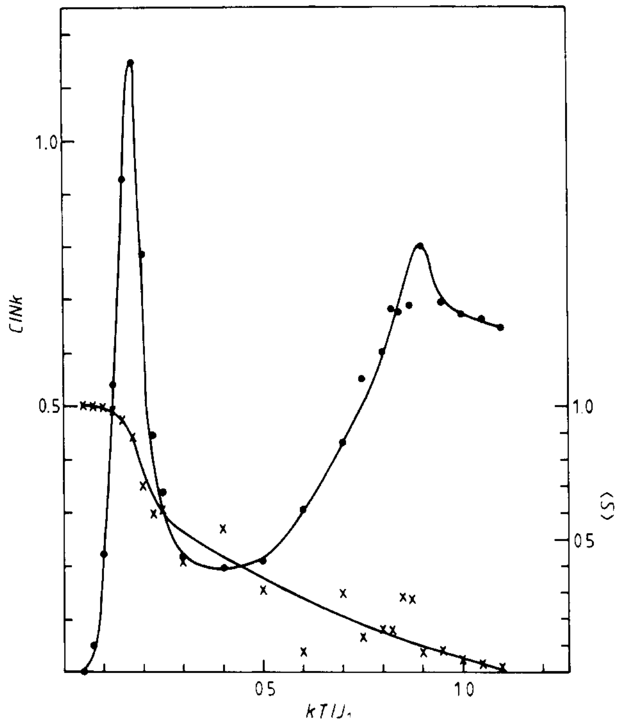


Figure 4. Temperature dependence of the specific heat C/Nk (●) and the order parameter $\langle S\rangle$ (×) for $J_2/J_1 = -0.3$. Full curves are guides to the eye.

In order to check if the intermediate phase indeed exists we calculated the spin–spin correlation function from our MC data in 54×54 lattices:

$$K(r) = |\langle S(\boldsymbol{n})S^*(\boldsymbol{n}+r\boldsymbol{\mu}_1)\rangle - \langle S(\boldsymbol{n})\rangle\langle S^*(\boldsymbol{n}+r\boldsymbol{\mu}_1)\rangle| \tag{9}$$

where $S(\boldsymbol{n}) = \exp(\mathrm{i}\theta(\boldsymbol{n}))$.

The results for $K(r)$ give only a qualitative indication whether the correlation length is of the order of the size of the system or of the order of a few lattice spacings.

Our results for $J_2/J_1 = -0.3$ are shown in figure 5.

For $kT/J_1 = 0.175$ (curve A, figure 5), which corresponds to a point inside the ferromagnetic phase (cf figure 4), $K(r)$ is short-ranged as expected. For $kT/J_1 = 0.30$ and 0.60 (curves B and C, figure 5) $K(r)$ falls off slowly with r, indicating that in this region of the phase diagram the correlation length is of the order of the size of our system. For $kT/J_1 = 0.80$ (curve D, figure 5) $K(r)$ falls with distance much faster than at $kT/J_1 = 0.30$ and 0.60, indicating that its behaviour is changing again to short-range, as expected in the high-temperature disordered phase.

These results confirm that there is an intermediate phase separating the low-temperature ferromagnetic phase from the high-temperature disordered one and suggest that it is a critical phase.

To investigate further the existence of a critical phase inside the ferromagnetic region, we analysed our MC data using the MCRG method outlined above. We show, in figure 6, a plot of $F_1(4, 16)$ and $F_1(4, 32)$ (cf equation (7)) for $J_2/J_1 = -0.30$ as a function of the temperature of the unrenormalised spin system. The bars in figure 6

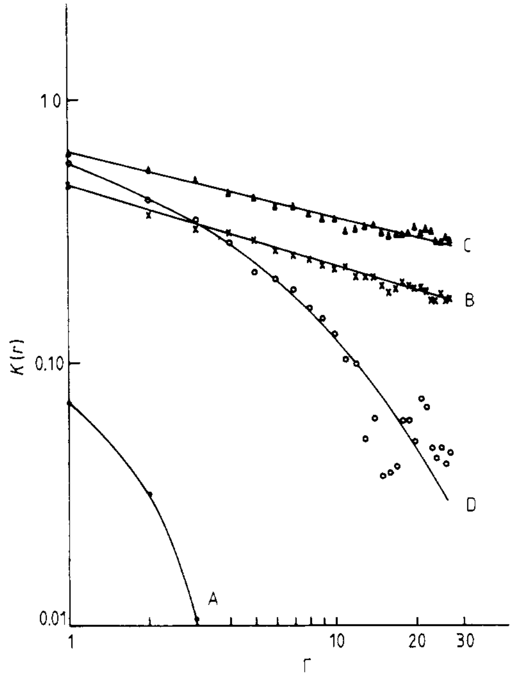


Figure 5. Variation with distance of the correlation function $K(r)$ for $J_2/J_1 = -0.3$. Full curves are guides to the eye. Values of kT/J_1: ●, 0.175; ×, 0.30; ▲, 0.60; ○, 0.80.

are estimates of the error associated with the value of the correlation function, using standard error analysis (Binder 1979). These two correlation functions are equal for $0.2 \leqslant kT/J_1 \leqslant 0.6$, indicating the existence of a critical phase between these two values. For $kT/J_1 < 0.2$, $F_1(4, 32) > F_1(4, 16)$, indicating (Tobochnik 1982) an ordered phase and for $kT/J_1 > 0.6$, $F_1(4, 32) < F_1(4, 16)$, indicating a disordered phase.

The location of the boundary between the ordered phase and the critical phase for $J_2/J_1 = -0.30$ obtained by MCRG coincides, within the errors of our simulation, with the location of the low-temperature specific heat peak (cf figure 4). The same is *not* true for the boundary between the critical phase and the disordered phase. In this case (for $J_2/J_1 = -0.30$) the MCRG locates the phase boundary at a lower temperature than that of the specific heat peak. For the five-state clock model ($J_2/J_1 = 0$) Tobochnik finds that neither of the two specific heat peaks coincide with the phase boundaries obtained by MCRG. This result is not surprising since in a transition from a critical phase to a disordered phase the specific heat does not have a singularity at the phase boundary, but shows a maximum above the transition temperature. As duality holds in the F region, there is no reason to expect a singularity in C/Nk at the boundary between the ferromagnetic phase and the critical phase either. At most, we expect to see a maximum in C/Nk associated with the transition. Thus the points where the specific heat has peaks do not coincide with the boundaries of the critical phase. The determination of these boundaries requires carrying out the MCRG analysis for several values of J_2/J_1, which we did not attempt. The available information from our own work and from Tobochnik (1982) is summarised in figure 10.

In the vicinity of the ferromagnetic five-state Potts model, where $J_2/J_1 = 1$ and $J_1 > 0$, our results are much more difficult to interpret. It is well known that the five-state

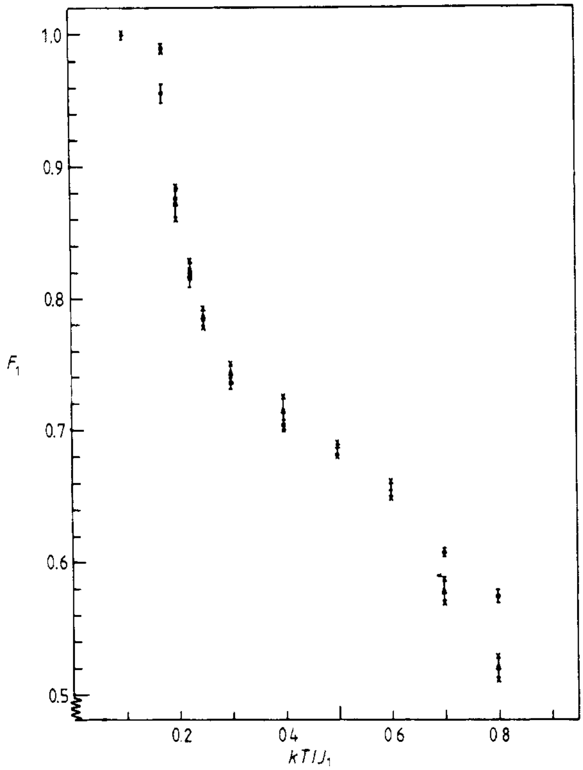


Figure 6. Variation of the correlation functions $F_1(4, 32)$ (▲) and $F_1(4, 16)$ (●) with the unrenormalised spin system temperature, for $J_2/J_1 = -0.3$.

ferromagnetic Potts model has a first-order transition at P (Baxter 1973). As $J_2/J_1 \to 1$ we observe that the two specific heat peaks obtained get closer and closer to each other until they merge into a single one located approximately on the self-dual line, SD, and that $\langle S \rangle$ vanishes faster and faster (figures 4 and 7). It has been suggested (Fateev and Zamolodchikov 1982) that the regions where the critical phase exist end at points on the self-dual line, SD, at each side of P (see figure 3), and that these points should correspond to models with $J_2/J_1 \simeq 0.64$ and $J_2/J_1 \approx 1.56$, which are exactly soluble. If this is true, the portion of the self-dual line between these two end points coincides with the phase boundary between the ferromagnetic and the disordered phases. The phase transition along this line is thought to be first order, as it is at P. The observation that the specific heat peaks merge at the self-dual line is consistent with this suggestion. However, we cannot determine precisely the endpoints of the critical phases nor can we determine the order of the transition near P.

Next we consider the AF region of the phase diagram. As we shall see in the discusssion that follows, our results indicate the existence of a critical phase at low temperature in the AF region and suggest that it is the continuation into this region of the critical phase observed in the F region.

The location of the specific heat peaks in the AF region is also shown in figure 3. We find that in MC runs with 10^4 MC steps/spin the specific heat has two maxima, similar to figure 4. As the number of MC steps/spin is increased to 6×10^4 MC steps/spin we verified, for a few values of J_2/J_1, that one of the maxima completely disappears

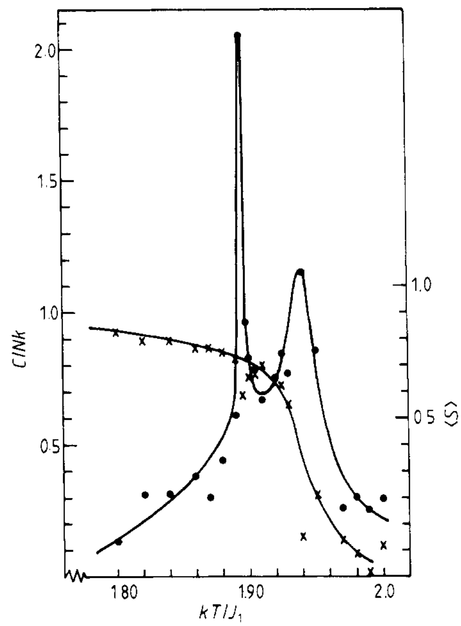


Figure 7. Temperature dependence of the specific heat C/Nk (●) and of the order parameter $\langle S\rangle$ (×) for $J_2/J_1 = 0.8$. Full curves are guides to the eye.

and the other, usually the high temperature one, remains unaltered, indicating the existence of a single transition. As we shall see below, the MCRG analysis reveals the presence, in the AF region, of only two different phases. We believe that the two-peak structure observed in this region using 10^4 MC steps/spin will change to a single peak if enough MC steps/spin are used. Thus, we interpret these results as indicating a single transition.

We also calculated from our MC data on 54×54 lattices the variation with distance of $K(r)$, equation (9). Some of the results are shown in figure 8. We see that $K(r)$ is long-ranged at low temperatures changing to short-ranged at higher temperatures, indicating also the existence of a single transition and suggesting that the low temperature phase is a critical one.

In order to further clarify the phase diagram in this region we carried out an analysis of the data using the MCRG method.

Some of our results using the $\lambda = 2$ or $\lambda = 3$ RG transformations (figures 1 and 2) are shown in figure 9. In figure 9(a) we plot the second neighbour correlation functions calculated using the $\lambda = 2$ RG transformation (figure 1), $G_2(4, 16)$, $G_2(4, 32)$ and $G_2(4, 64)$ as functions of the temperature of the unrenormalised spins systems, for $J_2/J_1 = -0.5$. We see that, within the errors of the simulation, these correlation functions are equal for $kT/J_1 \leqslant 0.45$ indicating a critical phase in this region. For $kT/J_1 > 0.45$ we find $G_2(4, 16) > G_2(4, 32) > G_2(4, 64)$, which is characteristic of a disordered phase. Therefore this analysis confirms the existence of a phase transition from the disordered phase at high temperature to a critical phase.

The results described above were qualitatively reproduced in calculations at different values of J_2/J_1, with other correlation functions and using both the $\lambda = 2$ and $\lambda = 3$ RG transformations.

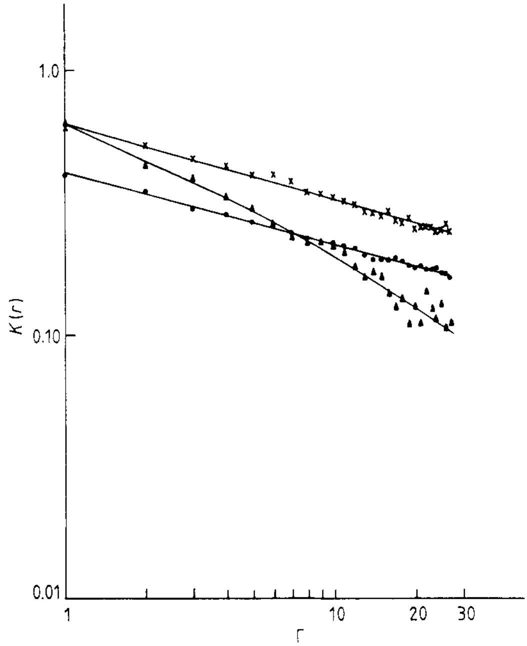


Figure 8. Variation with distance of the correlation function $K(r)$ for $J_2/J_1 = 0$. Full curves are guides to the eye. Values of kT/J_1: ●, −0.1; ×, −0.8; ▲, −1.0.

As examples we show, in figure 9(b) the correlation functions $F_2(8, 16)$ and $F_2(8, 32)$ ($\lambda = 2$ RG transformation) for $J_2/J_1 = -1.0$, in figure 9(c) $F_2(4, 16)$, $F_2(4, 32)$ and $F_2(4, 64)$ ($\lambda = 2$ RG transformation) for $J_2/J_1 = -1.4$ and in figure 9(d) $F_2(6, 18)$ and $F_2(6, 54)$ ($\lambda = 3$ RG transformation) for $J_2/J_1 = 0.5$ with $kT/J_1 < 0$.

For $J_2/J_1 = -1.0$ figure 9(b) and $kT/J_1 \leq 1.2$, the correlation functions $F_2(8, 16)$ and $F_2(8, 32)$ are equal within the statistical errors and for higher temperatures $F_2(8, 16)$ is greater than $F_2(8, 32)$. For $J_2/J_1 = -1.4$ (figure 9(c)) the three correlation functions $F_2(4, 16)$, $F_2(4, 32)$ and $F_2(4, 64)$ are equal for $kT/J_1 \leq 1.5$ and $F_2(4, 16) > F_2(4, 32) > F_2(4, 64)$ for $kT/J_1 > 1.5$. For $J_2/J_1 = 0.5$ figure 9(d) the $\lambda = 3$ MCRG results indicate that $F_2(6, 18)$ and $F_2(6, 54)$ are equal for $kT/J_1 \geq -0.6$ and that $F_2(6, 18) > F_2(6, 54)$ for $kT/J_1 < -0.6$. Thus, all three examples support the evidence of a phase transition from a high temperature disordered phase to a low temperature critical phase in the AF region. To check the consistency of our data, we compare in figures 9(e) and (f) the results obtained from the $\lambda = 2$ and $\lambda = 3$ RG transformations (figures 1 and 2) for the AF clock model. Both RG transformations predict, within the errors of the simulations, the same transition temperature $kT/J_1 \simeq -0.95$.

We also verified that the AF Potts model ($J_2/J_1 = 1.0$ with $J_1 < 0$) does not exhibit any phase transition, as shown in figure 9(g) where it is clearly seen that $F_2(8, 16) > F_2(8, 32)$ for all temperatures.

In the limit between the F and the AF regions, which corresponds to $J_2/J_1 = (\sqrt{5}-1)/(\sqrt{5}+1) \simeq -0.382$ and $J_1 > 0$ we performed a MCRG analysis and the behaviour of the correlation functions, shown in figure 9(h), indicates that this model has also a critical phase.

The results of the previous analysis are summarised in figure 10.

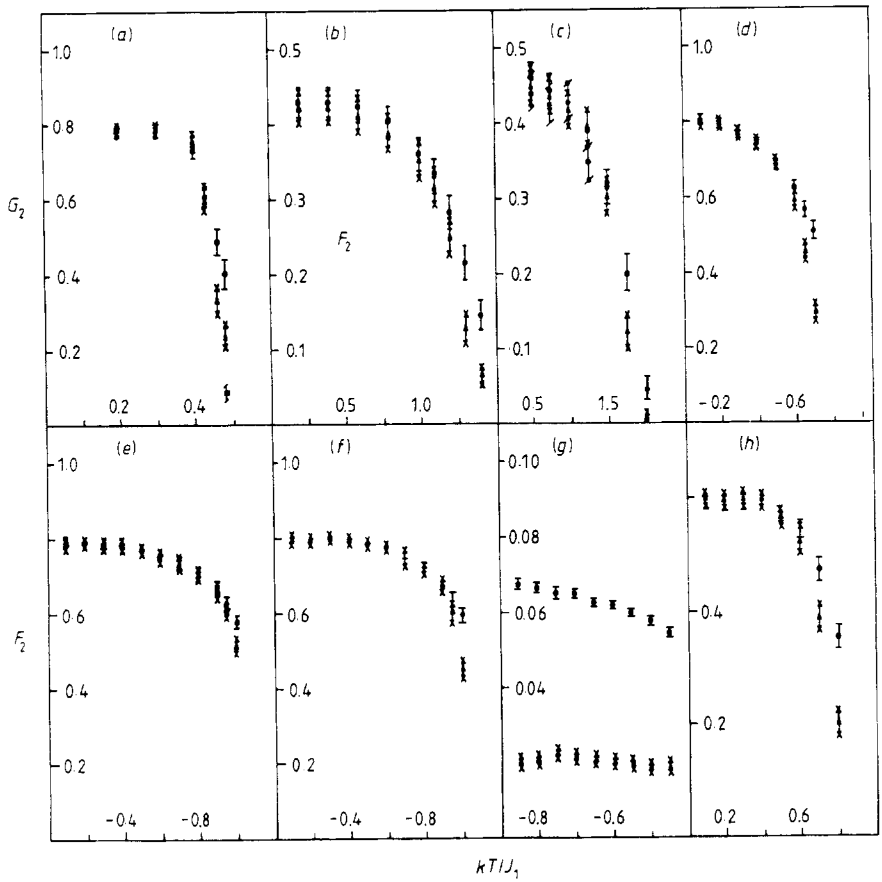


Figure 9. Variation of correlation functions with the unrenormalised spin system temperature for (*a*) $J_2/J_1 = -0.5$; ●, $G_2(4, 16)$; ▲, $G_2(4, 32)$; ■, $G_2(4, 64)$, (*b*) $J_2/J_1 = -1.0$; ●, $F_2(8, 16)$; ▲, $F_2(8, 32)$, (*c*) $J_2/J_1 = -1.4$; ●, $F_2(4, 16)$; ▲, $F_2(4, 32)$; ■, $F_2(4, 64)$, (*d*) $J_2/J_1 = 0.5$; ●, $F_2(6, 18)$; ▲, $F_2(6, 54)$, (*e*) AF clock model with $\lambda = 2$; ●, $F_2(4, 16)$; ▲, $F_2(4, 32)$, (*f*) AF clock model with $\lambda = 3$; ●, $F_2(6, 18)$; ▲, $F_2(6, 54)$, (*g*) AF scalar Potts model; ●, $F_2(8, 16)$; ▲, $F_2(8, 32)$, (*h*) $J_2/J_1 = -0.381\,97$; ●, $F_2(4, 16)$; ▲, $F_2(4, 32)$.

4. Conclusions

In the F region we confirm the existence of a critical phase. For $J_2/J_1 = -0.3$ this phase is identified qualitatively by the variation with distance of the spin-spin correlation function and by Wilson's MCRG procedure. This method also allows the determination of the transition temperature. These results are consistent with those obtained by Tobochnik (1982) for the five-state clock model ($J_2/J_1 = 0$). Our results, together with Tobochnik's, are shown in figure 10. Also shown are the duals to these points.

The phase boundaries joining these points are only suggestive. Two other points of the phase boundary are known exactly: the Potts point P and the Fateev and Zamolodchikov point Q($J_2/J_1 = 0.64$), both on the self-dual line. Following Fateev and Zamolodchikov's suggestion we draw the boundaries of the critical phase ending at Q.

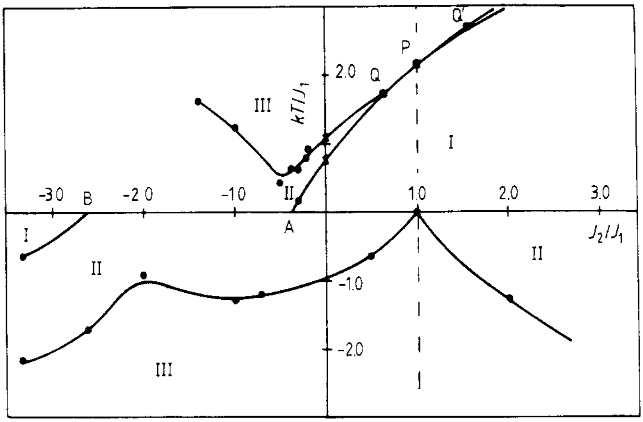


Figure 10. Suggested phase diagram for the $Z(5)$ model. (●) are our MCRG results and those obtained by symmetry, P is the transition point on the scalar Potts model line, Q is the transition point obtained by Fateev and Zamolodchikov, (×) are Tobochnik's MCRG results on the vector Potts model line $J_2/J_1 = 0$. Ferromagnetic ground states are located to the right of point A for $J_1 > 0$ and to the left of point B for $J_1 < 0$.

In the AF region we find two phases: a critical phase at low temperature and a disordered phase. The transition temperature for several values of J_2/J_1 has been determined by Wilson's MCRG method using two different RG transformations. The results are plotted in figure 10.

The behaviour of the spin–spin correlation function with distance confirms the existence of these two phases.

In figure 10 we draw the boundary between the critical and the disordered phases as a continuous line running from the F region into the AF region. This is, in our opinion, the simplest interpretation of our data and means that the critical phase in the F region penetrates into the AF region.

This interpretation is in disagreement with recent predictions made by den Nijs (1985) who studies the $Z(5)$ model and its excitations in the vicinity of the SOS model and finds two different critical phases, one in the F region and another one in the AF region. This means that the boundary between the critical phase in the F region and the disordered phase ends at $J_2/J_1 = -0.382$ and that another boundary between the critical phase in the AF region and the disordered phase begins at the same point. Our results for $J_2/J_1 = -0.382$ show that this model exhibits a critical phase.

In order to compare our results with those of den Nijs we plot our data in terms of the variables A and B defined as

$$A = \frac{1}{x_1 + x_2}$$

and

$$B = \frac{x_2 - x_1}{x_2 + x_1}$$

where x_1 and x_2 are given by equation (4).

In terms of these variables the self-dual line is $A=(\sqrt{5}+1)/2 \simeq 1.62$. The F region corresponds to $A>\frac{1}{2}$ and the AF region to $A<\frac{1}{2}$. The diagram is symmetric with respect to $B=0$. The $Z(5)$ model reduces to the SOS model for $B=\pm 1$ ($x_1=0$ or $x_2=0$) (Domany *et al* 1980).

Our results are shown in figure 11, together with the numerical results of Swendsen (1977) and Luck (1981). The predictions made by den Nijs are shown by broken lines in the same figure.

The disagreement with den Nijs' suggestion is evident from figure 11. As we pointed out before, the simplest interpretation of our results is that the critical phase in the F region penetrates into the AF region. However, the methods used in this paper cannot settle conclusively the exact nature of the phase boundary near the SOS model.

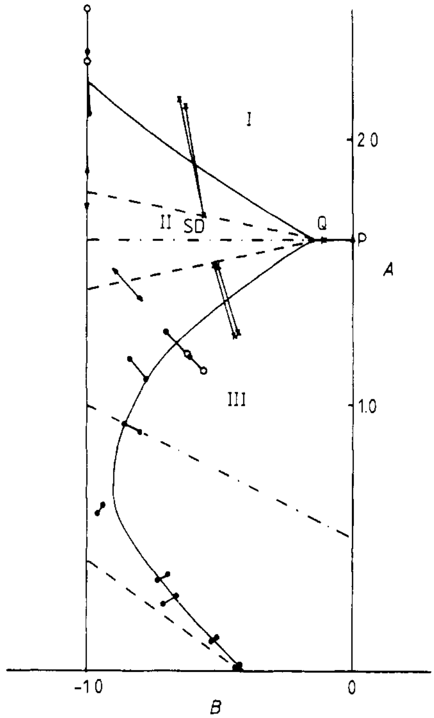


Figure 11. Suggested phase diagram for the $Z(5)$ model in the A and B variables. Bars represent errors. (●) are our MCRG results, (×) are Tobochnik's MCRG results and their duals, (▲) are Luck's results, (○) are Swendsen's results, P is the transition point of the scalar Potts model, Q is the transition point of the Fateev-Zamolodchikov model, broken lines are den Nijs' suggested phase diagram.

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References

Alcaraz F and Köberle R 1980 *J. Phys. A: Math. Gen.* **13** L153
Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
—— 1982 *Proc. R. Soc.* A **383** 43
Berker A N and Kadanoff L P 1980 *J. Phys. A: Math. Gen.* **13** L259
Binder K (ed) 1979 *Monte Carlo Methods* (Berlin: Springer)
Cardy J L 1981 *Phys. Rev.* B **24** 5128
Carneiro G M , Pol M E and Zagury N 1982 *Phys. Lett.* **92A** 258
den Nijs M P M, Nightingale M P and Schick M 1982 *Phys. Rev.* B **26** 2490
den Nijs M P M 1985 *Phys. Rev.* B **31** 266
Domany E, Mukamel D and Schwimmer A 1980 *J. Phys. A: Math. Gen.* **13** 311
Einhorn M, Savit R and Ravinovici R 1980 *Nucl. Phys.* B **170** 16
Elitzur S, Pearson R and Shigemitsu J 1979 *Phys. Rev.* D **19** 3698
Fateev V A and Zamolodchikov A B 1982 *Phys. Lett.* **92A** 37
Luck J M 1981 *J. Physique Lett.* **42** L275
Niemeijer Th and van Leeuwen J M J 1976 *Phase Transitions and Critical Phenomena* vol 6 (New York: Academic) p 425
Roomany H and Wyld H W 1981 *Phys. Rev.* B **23** 1357
Rujan P, Williams G O, Frisch H L and Forgács G 1981 *Phys. Rev.* B **23** 1362
Swendsen R H 1977 *Phys. Rev.* B **15** 5421
—— 1982 *Real Space Renormalization* (Berlin: Springer) p 57
Tobochnik J 1982 *Phys. Rev.* B **26** 6201
Wilson K unpublished